Assessing a Modeling Process of a Linear Pattern Task

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This research investigated the process of generalizing a pictorial linear pattern problem, as done by fifty-three mathematically promising students participating in an after school math club. The students' work revealed a range of solution paths and representations, and a cycle of expressing – testing – revising. While the majority of them found the constant difference property of the pattern, they experienced difficulties in expressing the general rule. The majority of students applied recursive strategies, even when more global strategies were called for. Although the aforementioned task lacks a real-life context that is essential for modelling problems, the advantages of such problems in multi-cultural classes are discussed.

THEORETICAL BACKGROUND

Mathematical modeling problems are defined as problems that involve mathematizing objects, relations, operations, patterns, and regularities. Model-eliciting activities are characterized by their complex real-life context, and solving them engages students in a cycle of "expressing – testing – revising" (Lesh & Doerr, 2003; Lesh, Yoon & Zawojewski, 2006).

Krutetskii (1976) claimed that mathematical giftedness consists of several abilities, including pattern recognition, the ability to generalize, the ability to reason, and the flexibility of mental processes in mathematical activities.

The process of generalizing is a crucial aspect of mathematical thinking, especially higher order mathematical thinking (Sriraman, 2004), and, of course, in mathematical modeling (Lesh & Doerr, 2003). (Note: there is a large theoretical foundation on generalization (e.g. Dorfler, 1991; Harel & Tall, 1991), which is beyond the scope of this paper). In order to generalize correctly one has to single out similarities in structures and relationships (Krutetskii, 1976).

Components of mathematical ability and the generalization process can be revealed in pattern problems. Linear patterns are patterns in which the n*th* element can be expressed as $an + b$. They can be presented in a range of representations, including figural or numerical (Zazkis & Liljedahl, 2002). These pattern problems have been found to be challenging for middle school students (English & Warren, 1998; Stacey, 1989).

There are four main strategies that correspond to the cognitive levels of mathematical development: *Procedural activity* – at this level the students recognize the constant difference of the linear pattern, and are focused on the procedural feature of the pattern, i.e. adding the constant difference. This is the level where students can successfully perform "near generalizations", that is, obtain a correct answer in a step by step approach, either by drawing or by calculating (Garcia-Cruz & Martinon, 1998; Stacey, 1989). *Procedural understanding* - students are able to establish an invariant from the figures and to apply the calculation rule. This stage is characterized by the students' ability to verbally express a rule (Garcia-Cruz & Martinon, 1998). *Searching for a functional relationship* - students at this stage can see the global structure of the pattern, can obtain a solution for "far generalizations", and can express the patterns' rule in a non-formal way. The fourth level involves not only understanding the *functional relationship* but also expressing it in a *formal algebraic* way.

This research investigated how mathematically promising middle-school students solve linear pictorial pattern problems. Although not a real-life situation, it offers insight into the generalization development, which is crucial to all modeling activities.

METHODOLOGY

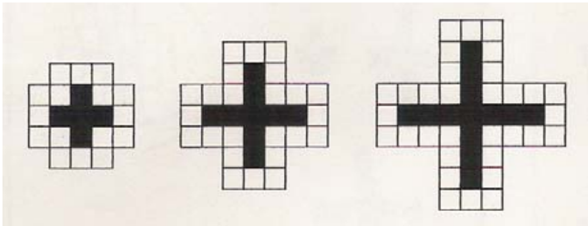
Population

The research population was made up of fifty-three mathematically promising middle school students who participate in "Kidumatica" - an after-school math club in southern Israel. Kidumatica offers a variety of mathematical activities for above average students. The students participating in Kidumatica come from diverse backgrounds – some are Israeli natives and some are immigrants from various countries. Some live in urban communities and others in villages, kibbutzim etc.

Settings and Research Instrument

The research instrument was a questionnaire comprised of six non-routine problems, including the pattern task discussed in this report. The task (fig. 1) was based on the research of Rivera & Becker (2005), and contained four graduated questions dealing with a pictorial linear sequence. In their research, Rivera and Becker investigated the figural versus numerical modes of generalization of prospective school teachers. In this research, we used the same task to investigate the generalization process of middle school students participating in the math club. The students were specifically instructed to describe their solution path.

The following illustration presents the three first patterns in a sequence:



A. How many white tiles are needed to make the next pattern?

B. How many white tiles are needed to make pattern 10?

C. Suggest a method to calculate the number of white tiles needed to make any pattern in this sequence.

D. Suggest a method to calculate the number of white tiles needed to make the n^{th} pattern in this sequence

Fig. 1: The pattern task

Although the students had a sufficient background to meet the challenge, the problem was considered non-routine because pattern tasks do not appear in the standard textbooks. Also important to note is that most of the students participating in this research did not have formal instruction in algebra.

Questions A and B could be solved in a step by step approach. Question A (the immediate pattern), was a "warm up" question, that enabled the solvers to examine and investigate the pattern. Obtaining a correct answer could be achieved either by drawing the next pattern, or by grasping the patterns' general rule.

Question B demanded pattern recognition as well as forming a generalization, although a correct answer could be obtained simply by adding up the numbers, or even by drawing.

Questions C & D dealt with pattern generalization. Question C enabled the students to represent the generalization in any form with which they felt comfortable. In Question D there was an explicit demand that the generalization be presented in a formal algebraic mode. These two questions provided the distinction between those students who can only "think algebraically" and those who can also "write algebraically".

Analysis Methods of Student Answers

The answers were analyzed qualitatively according to three criteria (Neria & Amit, 2004; 2006): correctness of the answers, solution strategy, and the mode of representation in the solution path. (Note: our paper omits the third category).

Correctness of answers: The analysis included the right answer, the wrong answer, and no answer. This category referred only to the final answer, regardless of solution paths.

Solution strategy / method: The categories were based on previous studies (English & Warren, 1998: Ishida, 1997: Lee, 1996) and are described in table 1. The two forms of the additive strategy (drawing or counting) are non-general methods. In contrast, searching for a functional relationship (from the figures or from a table/list) are general methods.

Strategies	***Sub-Categories***	***Example***
Additive strategy (recursive approach): Students observed that each step increases by a constant difference	I) Drawing a figure and counting to get an answer.	Fig. 2
	II) Setting up a table/list and completing it to get to the requested answer.	Figs. 2, 3, 4
Searching for the functional relationship: Students attempt to identify the function that describes the pattern		Figs. 5, 6

Table1: Students' Solution Strategies for Solving Near-Generalization Problems

RESULTS

Correctness of answers: 32 students (60.4%) got the correct answer for Question A, 24 students (45.3%) were correct in questions B and C. Only 3 students succeeded in writing the correct pattern rule in a formal algebraic form. (See Table 2)

	Question A The next pattern	Question B The 10th pattern	Question C Intuitive generalization	Question D Algebraic generalization
Correct Answer	60.4%	45.3%	45.3%	5.7%
Wrong Answer	33.9%	45.3%	35.8%	41.5%
No Answer	5.7%	9.4%	18.9%	52.8%
Total	100%	100%	100%	100%

Table 2: Distribution of the correctness of answers (N=53)

Solution strategies: In the questionnaire, the students were specifically instructed to describe their solution path. This led to rich data in words, numbers and diagrams. Table 3 refers to the distribution of the strategies used in Questions A, B and C. The majority of students employed additive strategies that usually entailed forming a table or a list (see figs. 2, 3, 4). A minority of students employed a functional strategy. They searched for the sequence rule and applied it (see fig. 5 for application of the correct rule, and fig.6 for application of an incorrect rule).

	Question A The next pattern	Question B The 10th pattern	Question C Intuitive generalization
Additive	77.3%	64.2%	49.0%
Functional	7.6%	15.1%	30.2%
No solution path demonstrated	9.4%	11.3%	1.9%
No Answer	5.7%	9.4%	18.9%
Total	100%	100%	100%

Table 3: Distribution of the strategies (N=53)

DISCUSSION

As seen in the results, 60% of the students knew how to calculate the number of tiles needed for the next pattern. 45% knew how to calculate the number of tiles needed for the 10th pattern (near generalization) and how to express the generalization intuitively. Far less (6%) succeeded in generalizing in a formal algebraic form.

It was no surprise that the students had difficulties in forming an exact algebraic expression, since most of them had not begun their algebra studies. As in former studies (English & Warren, 1998; Lee, 1996), the constant difference property was usually recognized, enabling students to find the n^{th} element of a pattern from the *(n-1)*th element. However, the attempt to generalize was found to be difficult, even when generalizing in informal modes. Students engaged concrete strategies where more global strategies were called for, and preferred recursive approaches to the functional ones. The occurrence of fixation of a recursive approach found in this study supports previous researches (English & Warren, 1998; Lee, 1996).

The additive method was recognized and applied correctly by most of the students in the near generalization. Those who failed to generalize tended to start out with additive strategies, but it's possible they lacked the flexibility to switch to global strategies needed for reaching generalizations (English & Warren, 1998; Garcia-Cruz & Martinon, 1998; Zazkis & Liljedahl, 2002).

Another possible explanation for the difficulties in generalizing is a phenomenon described by Rivera & Becker (2005). In their study, they found that some students are predominantly more numerical than figural, causing difficulties in employing visual strategies. The predominantly figural students employ visual strategies and manage to focus on identifying relationships toward forming a generalization, while the predominantly numerical ones lack the flexibility to tackle the figural patterns.

Although generalizing is a basic human activity and is demonstrated in young children (Lee, 1996), the type of generalizing needed while solving pattern problems generated several difficulties at different stages: at the perceptual level (recognizing the pattern), the verbal level (expressing the pattern) and at the symbolization level (using *n* to represent the *n*th pattern in a sequence) (Lee, 1996).

The ability to continue a pattern comes well before the ability to describe the general term. However, recognizing a pattern doesn't necessarily lead to a generalization (Zazkis & Liljedahl, 2002), and verbalizing a generalization doesn't necessarily lead to a generalization in algebraic form.

It is reasonable to assume that students using functional strategies have the basis to develop algebraic thinking. On the other hand, one can't assume that those who did not use functional strategies do not have the basis and potential for developing algebraic thinking, mainly because the problem is not sophisticated enough.

Did the students go through a modeling process? If we take the rigorous, somehow narrow modeling definition – then no. While the problem has some of the elements of modeling, such as a range of solution paths and representations, and students experiencing a cycle of expressing – testing – revising, the problem lacks some of the essential characteristics of model eliciting problems, such as real life context or no single correct solution (Lesh & Doerr, 2003).

Taking a wider perspective on modeling problems, particularly on the "*real-life context*," raises a major question to consider – what kind of problems can be taken as "*real-life context*" and be meaningful for a wide range of students?

Real life situations are dependant on culture and environment, so if we take into account multi-cultural populations, what kind of problems can be meaningful for <u>all</u> students participating in a math class? The problem presented in this paper can be perceived as a prototype of a recursive activity towards generalizations of linear patterns. The problems' lack of a traditional real life context can be taken as an advantage, particularly when dealing with multi-cultural classes, and can increase the accessibility of model eliciting activities.

This "tension" between the rigorous narrow stand of "real life" and the wider meanings that can incorporate more types of problem solving requires further discussion and elaboration.

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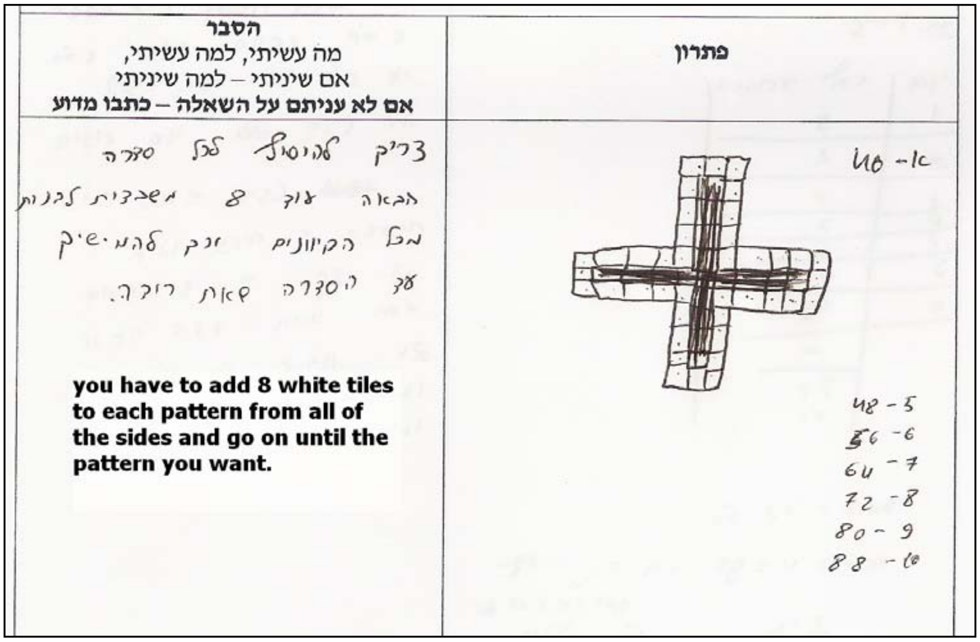
הסבר
מה עשיתי, למה עשיתי,
אם שיניתי – למה שיניתי
אם לא עניתם על השאלה – כתבו מדוע
פתרון
you have to add 8 white tiles to each pattern from all of the sides and go on until the pattern you want.

Fig. 2

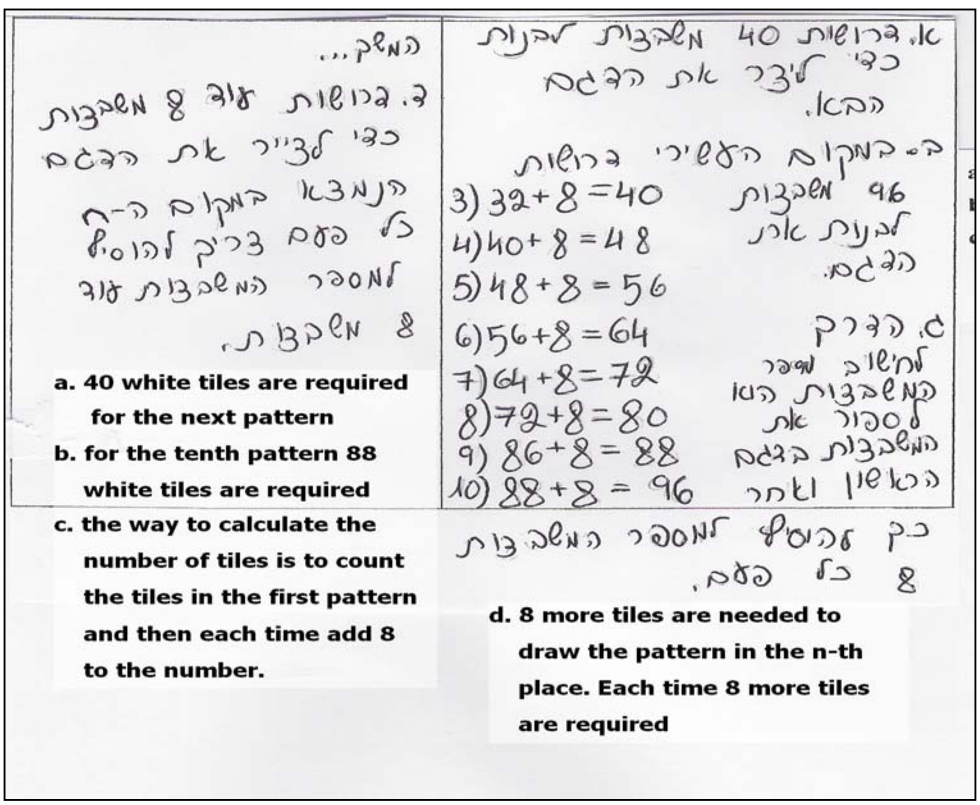
a. 40 white tiles are required for the next pattern
b. for the tenth pattern 88 white tiles are required
c. the way to calculate the number of tiles is to count the tiles in the first pattern and then each time add 8 to the number.
d. 8 more tiles are needed to draw the pattern in the n-th place. Each time 8 more tiles are required

Fig. 3

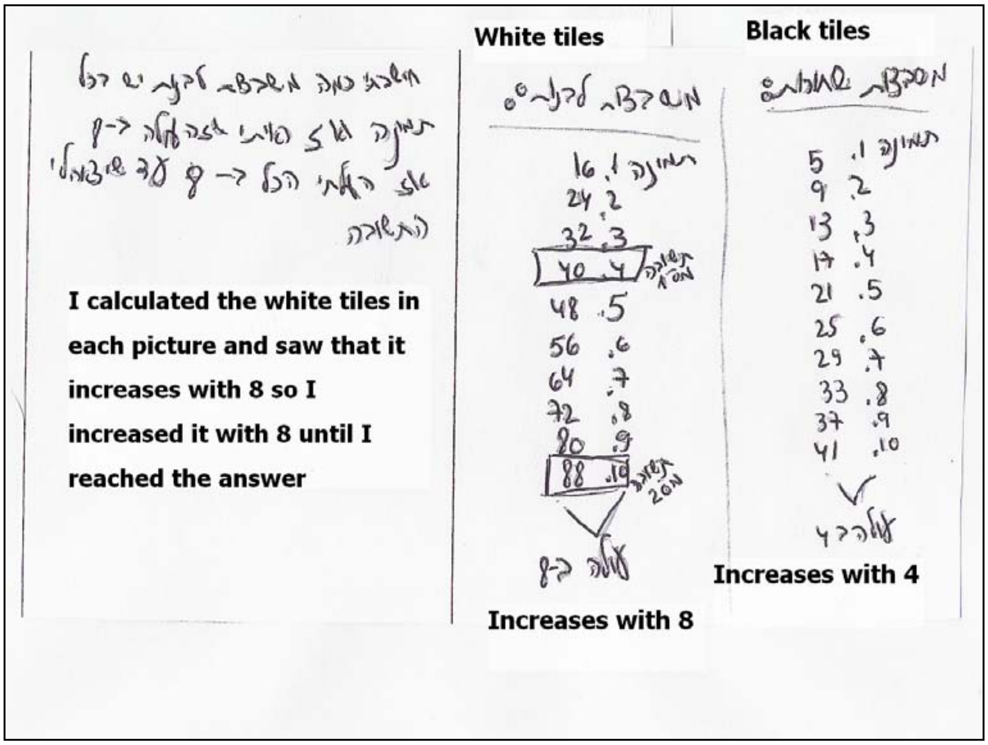
White tiles
Black tiles
I calculated the white tiles in each picture and saw that it increases with 8 so I increased it with 8 until I reached the answer
Increases with 8
Increases with 4

Fig. 4

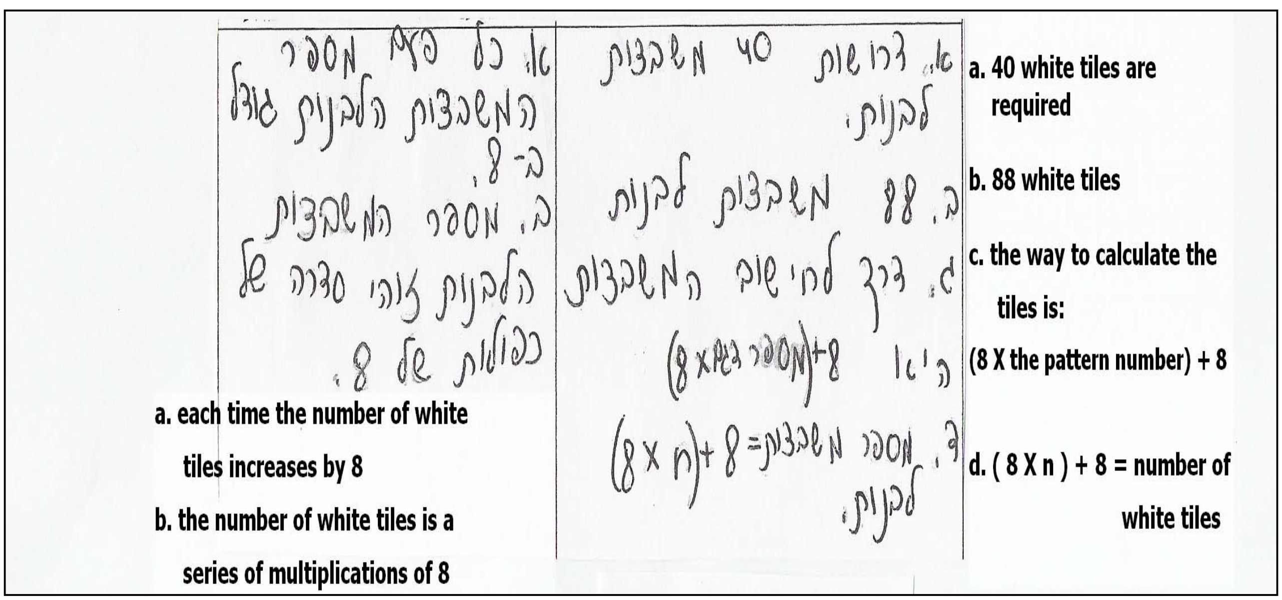
a. 40 white tiles are required
b. 88 white tiles
c. the way to calculate the tiles is:
(8 X the pattern number) + 8
d. (8 X n) + 8 = number of white tiles
a. each time the number of white tiles increases by 8
b. the number of white tiles is a series of multiplications of 8

Fig. 5

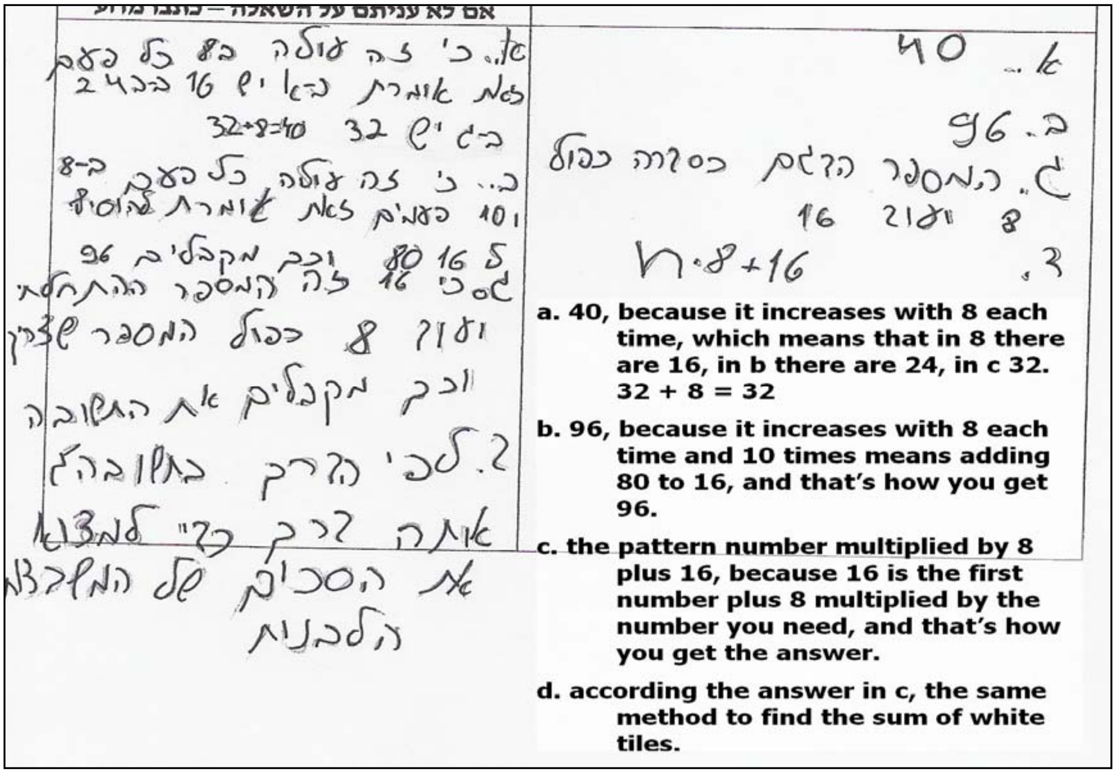
a. 40, because it increases with 8 each time, which means that in 8 there are 16, in b there are 24, in c 32. 32 + 8 = 32
b. 96, because it increases with 8 each time and 10 times means adding 80 to 16, and that's how you get 96.
c. the pattern number multiplied by 8 plus 16, because 16 is the first number plus 8 multiplied by the number you need, and that's how you get the answer.
d. according the answer in c, the same method to find the sum of white tiles.

Fig. 6