

# **Modeling as Isomorphism: Using New Technologies in Mathematics Teacher Education**

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*ABSTRACT: This paper presents modeling activities designed for technology-enhanced mathematics teacher education courses aimed at the development of grade-appropriate mathematical concepts. Conceptualized in terms of isomorphism, these activities deal with various problem-solving topics in arithmetic, algebra, geometry, and discrete mathematics. The paper reflects work done with prospective K-12 teachers and it includes their reflections on the use of technology in modeling.*

## **Introduction**

The paper shares pedagogical ideas that emphasize the importance of teaching mathematics to prospective K-12 teachers (referred to below as teachers) through modeling with new technologies in a variety of problem-solving contexts. The very notion of modeling is discussed in terms of isomorphism. This abstract concept, applied “when two complex structures can be mapped onto each other, in such a way that to each part of one structure there is a corresponding part in the other structure” (Hofstadter, 1999, p. 49), has rather concrete meanings and can serve as a conceptual underpinning of mathematical modeling. Indeed, the latter can be described as a process of exploring properties of objects (contextual situations) that belong to one system by mapping them onto another system that is structurally identical (or isomorphic) to it. Through this process, one creates a model (either physical or symbolic), explores its properties, develops methods of investigation, comes up with meaningful results, and then *interprets* these results in the language of the original system.

Typically, a model used to describe an object (situation), is simpler than the object itself. Sometimes, however, a model can be at least as complex as an object; yet the model enables for the development of methods of investigation that are not available within a system to which the object belongs. The paper demonstrates how typical, standards-based problem-solving situations from the K-12 curricula, can be associated with elementary-but-powerful models that enable one to develop the appreciation of how mathematics can be approached, “at least initially, ... from an experientially based direction rather than an abstract/deductive one” (Conference Board of the Mathematical Sciences, p. 96). It shows that modeling pedagogy has the potential to develop an *expertise* in bridging seemingly disconnected structures, whereas success of that pedagogy depends on one’s ability to interpret (possibly hidden) isomorphic relationships between the model and the object (situation) it describes.

More specifically, the paper analyses various modeling activities designed for technology-enhanced mathematics teacher education courses taught by the author and used both as learning and practice-oriented tools. These activities cover problem-solving topics in arithmetic, algebra, geometry, and discrete mathematics. The paper reflects work done with the teachers within the courses aimed at the development of grade-appropriate mathematical concepts through modeling and applications. Technology is understood broadly in terms of cultural tools (Cobb, 1995) to include learning

environments utilizing a spreadsheet, dynamic geometry program, graphics software, as well as concrete materials. The paper shows various ways of using these tools in demonstrating isomorphic relationships between objects under study and their respective models.

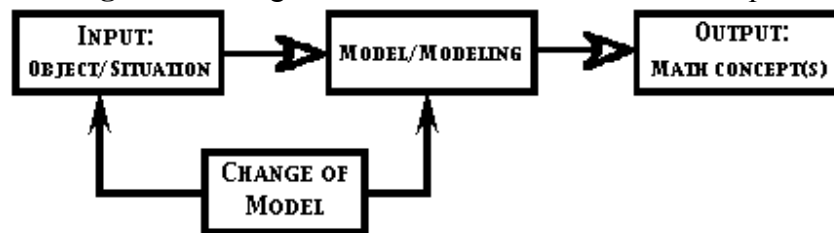
Moreover, it is the use of these tools that makes mathematics learning possible in a way that has not been available in the absence of these tools (Noss & Hoyles, 1996; Johnson & Lesh, 2003). Teachers' reflections on modeling activities as well as models developed and interpreted by young children in the field are included in the paper. These reflections suggest that modeling activities discussed in the paper contribute to the preparation of qualified teachers of mathematics, something being increasingly considered "the best way to raise [average] student achievement" (Conference Board of the Mathematical Sciences, 2001, p. 3).

**Figure 1.** Model does not change unless input changes



Another aspect of modeling discussed in the paper concerns two ways in which isomorphic relationships can be extended. One way is to change the structure of the object (situation), develop a new model, and refine the corresponding methods of exploring the model. Another way is to change the structure of model first, and then change the object (situation), develop new results, and then see how to modify the object (situation) in order to make new results applicable to it. In other words, the first approach (Figure 1) describes the case of model being dependent on object but not vice versa; the second approach (Figure 2) describes the case when contextual inquiry results from the meaningful change of parameters of a model, a process requiring a higher level of mathematical thinking.

**Figure 2.** Change of model within a model affects input



### Modeling with Counters

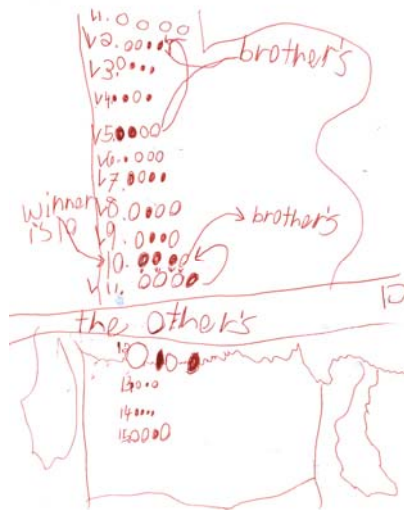
The use of concrete materials (or their electronic analogs) at the elementary level is commonly viewed as an important precursor to the formal learning of mathematical concepts. This hands-on approach to mathematics *is* modeling. Indeed, by acting on manipulatives in a teacher-guided learning environment, one can create a physical model that is structurally isomorphic to an intended symbolic representation of a mathematical situation. Once the model is constructed, it can be interpreted symbolically in the form of numerical (or, more generally, algebraic) relationships. The illustrations of this section

show how modeling with counters followed by the appropriate interpretation of a model can lead to the development of rather sophisticated mathematical concepts.

### Illustration 1. Partition of integers into summands

Consider the following problem: Find all ways to represent a positive integer  $n$  as a sum of counting numbers. This mathematical situation can be modeled using two-color counters as suggested in the following task adapted from New York State mathematics core curriculum (New York State Education Department, 1998): *Using four two-color counters, create different combinations of red and yellow. Record each combination found and stop when you believe you found all of them.*

**Figure 3.** A model constructed by a second grader



This task was given to a second grade pupil of average mathematical abilities. The pupil's response is shown in Figure 3. It represents an (almost complete) physical model constructed through trial and error that is short of one combination only. However, a true intellectual accomplishment of the pupil is manifested by his reflective analysis of the model; particularly, by the use of the term "brothers." In this, the pupil really attempted to *interpret* the model. From this interpretation, a generalized conceptual system (English, 2003) can emerge. The development of such a system is the main purpose of using concrete materials as modeling tools. In fact, a hidden meaning of the task is to create conditions for generalization by moving from a physical model to its symbolic representation. By interpreting each combination of counters numerically, one can develop understanding of what changes and what remains invariant when the number of counters involved becomes a variable quantity.

For example, the second combination in Figure 3 can be described in terms of the action on counters as follows: two yellow counters when added to two red counters give the total of four counters. In a decontextualized form, this verbal description can be represented numerically as  $2+2=4$ . But the fifth combination in Figure 3, being associated by the pupil with the second combination, has the same numerical meaning! By pairing

“brothers,” one puts all red-yellow combinations of four counters in eight groups each of which has unique numerical interpretation. One can see that the physical model (almost completely) constructed by a second grader is structurally isomorphic to eight representations of four as a sum of counting numbers with regard to the order of summands ( $4=1+1+1+1$ ,  $4=1+1+2$ ,  $4=1+2+1$ ,  $4=2+1+1$ ,  $4=1+3$ ,  $4=3+1$ ,  $4=2+2$ ,  $4=4$ ).

The development of generalized conceptual system can be enhanced through the discussion of two ways in which modeling activities with counters can be extended. As mentioned at the beginning of this section, one can be asked to find all ways to represent a positive integer  $n$  as a sum of other counting numbers. Through such extension, contextual inquiry changes, yet the model essentially remains the same (Figure 1). By using recursive reasoning, teachers can prove that the number of combinations of red and yellow doubles as one moves from  $k$  to  $k+1$  counters. This requires a reorganization of a model and then interpretation of this reorganization in terms of relationships between two problematic situations. As a result, in the output one would get  $2^{n-1}$  pairs of “brothers.” In other words, one can conclude there exists  $2^{n-1}$  different ways to represent  $n$  as a sum of counting numbers.

### **Illustration 2. Fibonacci numbers**

As another (Figure 1-type) extension of the above situation, one can be asked to *create different size samples of two-sided (red and yellow) counters in which no two red counters appear in a sequel*. In doing so, one can discover that there are two ways to create a one-counter sample, three ways to create a two-counter sample, and five ways to create a three-counter sample. Furthermore, by interpreting model so constructed, one can put all possible combinations that make the four-counter sample in two distinct groups. In the first group, the first counter is yellow – the total number of possible arrangements of that type equals to the total number of elements in the three-counter sample. In the second group, the first counter is red – the total number of possible arrangements of that type equals to the total number of elements in the two-counter sample. Through this activity, prospective teachers can be introduced to Fibonacci numbers in which every number beginning from the third equals to the sum of the previous two. In other words, there exists an isomorphism between Fibonacci numbers and the counter-based physical model through which these numbers can be discovered. Once again, it is through the interpretation of a hands-on activity that a new conceptual system, a recursion, can be developed.

### **Illustration 3. Combinations without and with repetitions**

Another way to extend modeling with counters described above) is to change the model first and then find a matching context (Figure 2). This can be done in a number of ways also. For example, one can be asked to *find all possible arrangements of two red and three yellow (one-sided) counters*. This activity represents a change of model (over the original one), as the number of counters in each color is fixed and cannot be changed through manipulation. In finding all ways of creating such combinations, one develops a model for finding all three-combination of five counters, or, more generally, all  $m$ -combination of  $n$  objects. One can show that in this case, there exists an isomorphism between the five-counter model and a five-letter word  $YYN$ . The letter  $Y$  means that the corresponding object is included in the combination; the letter  $N$  means that the

corresponding object is not included in the combination. Such change of model can bring about a new context. This context is associated with a basic combinatorial problem – finding all three-combination of five objects. It (context) can be extended further to allow for the introduction of combinations with repetitions through the variation of model. Finally, from both models generalized conceptual systems can be developed.

#### **Illustration 4. Modeling with Graphics Software**

Modeling activities based on the notion of isomorphism can be carried out with young children using graphics software Kid Pix Studio Deluxe (Broderbund Software, Inc., 2001). These activities show how teachers can help the children develop early algebra skills by mapping a didactically coherent (Abramovich & Cho, 2007) word problem – typically requiring the use of equations – into its iconic representation and then interpret symbolically a physical model so constructed using the mediating power of “talking” letters and numerals.

An interesting observation resulting from the modeling activities of that kind is that the children are able to move with relative ease from a physical model to its representation through written symbols. A typical approach to a word problem is to start with writing equations (a mathematical model) and continue with solving these equations using strictly defined syntactic actions on a notation system. Alternatively, the children can be encouraged to move in the opposite direction by being inducted into the use of symbolism through modeling with the graphics software.

As mentioned elsewhere (Abramovich, 2005), this alternative approach provides young children with tools that can mediate their transition from using “first-order symbols ... directly denoting objects of actions [to using] second-order symbolism, which involves the creation of written signs for the spoken symbols of words ... [and] develops [by] shifting from drawing of things to drawing of words” (Vygotsky, 1978, p. 115). Through a drawing-based modeling young children can move from iconic models of problematic situations to writing algebraic equations that rigorously describe the situations. A practical implication of teaching second-order symbolism through modeling with graphics software is that, like reading skills (Vygotsky, 1978), the children can develop algebraic skills earlier than usual.

#### **Modeling with Spreadsheets**

This section describes modeling activities used by the author with prospective teachers of secondary mathematics. The *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics, 2000, p. 335) includes an interesting example of a problem-solving activity recommended for grades 9-12 that can be interpreted in terms of isomorphism. It shows how the  $n \times n$  multiplication table and a checkerboard of the same size can be mapped onto each other so that the former (numeric) structure can be utilized in counting all possible rectangles on the latter (geometric) structure. In other words, the multiplication table serves as a mathematical model for a counting problem in plane geometry. The following illustration shows how extending this counting problem/situation (i.e., changing input) may bring about a new model from which one can develop new concepts.

### Illustration 5. Counting rectangles with special properties

On a square size checkerboard find the total number of rectangles at least one side of which is an odd (even) number.

This extension represents the change in situation that requires the change of model. The existing model – the  $n \times n$  multiplication table – should be modified to reflect the absence of rectangles with both sides having even (odd) dimensions. First, consider the former case on a square checkerboard of even size  $n=2l$ . All rectangles with two even dimensions can be put into  $l^2$  groups so that rectangle of the size  $2k \times 2m$  ( $k, m = 1, 2, \dots, l$ ) is associated with an odd product  $(n-2k+1)(n-2m+1)$  in the  $n \times n$  multiplication table. For example, when  $n=10$ ,  $k=1$ ,  $m=2$ , one has a rectangle of the size  $2 \times 4$  associated with the cell containing the product  $7 \times 9$  (in that order). In other words, all rectangles of the size  $2k \times 2m$  ( $k, m = 1, 2, \dots, l$ ) can be mapped onto the cell  $(n-2k+1)(n-2m+1)$  in the  $n \times n$  multiplication table. Therefore, the total number of rectangles with two even dimensions on an even size square checkerboard equals to the sum of all odd numbers in the multiplication table of that size.

In finding this sum, one can use a spreadsheet. Figure 5 displays all odd products in the  $10 \times 10$  multiplication table. It suggests that the sum of the numbers displayed equals to  $(1+3+\dots+9)+3(1+3+\dots+9)+\dots+9(1+3+\dots+9) = (1+3+\dots+9)^2 = (5^2)^2 = 625$ . In general, the sum,  $S_{\text{odd}}(2l)$ , of odd numbers in the  $2l \times 2l$  multiplication table

$$S_{\text{odd}}(2l) = (1+3+5+\dots+2l-1)^2 = (l^2)^2 = l^4.$$

Finally, by substituting  $n=2l$  in formula (1), one can find  $S_{\text{even}}(2l)$  – the sum of all even numbers in the  $2l \times 2l$  multiplication table – as follows

$$S_{\text{even}}(2l) = (1+2+3+\dots+2l)^2 - l^4 = l^2(l+1)(2l+1).$$

This sum equals to the number of rectangles with at least one odd dimension on the  $2l \times 2l$  checkerboard.

**Figure 4.**  $2 \times 4$  and  $4 \times 6$  rectangles associated with products  $9 \times 7$  and  $7 \times 5$  respectively

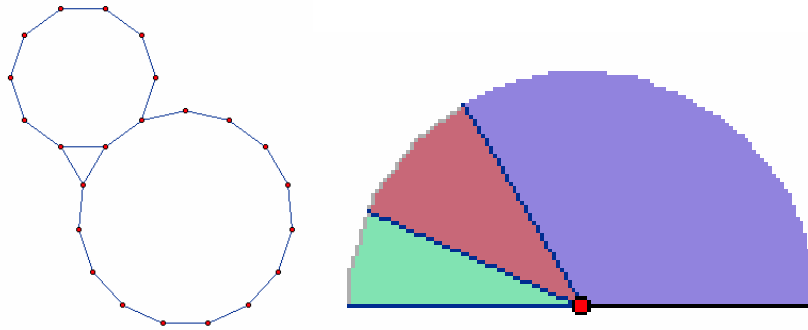
X	1	2	3	4	5	6	7	8	9	10
1	1		3		5		7		9	
2										
3	3		9		15		21		27	
4										
5	5		15		25		35		45	
6										
7	7		21		35		49		63	
8										
9	9		27		45		63		81	
10										

In a similar way, one can establish an isomorphism between rectangles with at least one even dimension on a square checkerboard of odd size and the cells of a corresponding multiplication table. Note that one can also count rectangles with at least one odd dimension on an odd size checkerboard as well as rectangles with at least one even dimension on an even size checkerboard (Abramovich, 2007). This would require somewhat more complicated counting techniques. However, the idea of modeling as isomorphism and spreadsheet applications remain unchanged for these more complex considerations.

### Illustration 6. Modeling with *The Geometer's Sketchpad*

Modeling activities can be carried out by teachers in a geometric context of tessellations defined as the process of tiling the plane with no gaps or overlaps by using one or more shapes. Typically, these shapes are regular polygons. In this section, the edge-to-edge tessellation – covering the space around a point by using one or more regular polygons – will be of special interest. For example, the triple of regular polygons with 3, 10 and 15 sides can be put together to cover the space around a point at which their vertices meet (Figure 5, left).

**Figure 5.** Edge-to-edge tessellation (left) and its isomorphic model (right)



One can check to see that  $60^\circ + 144^\circ + 156^\circ = 360^\circ$ . One can further verify that  $1/3 + 1/10 + 1/15 = 1/2$ ; i.e., the sums of reciprocals of the number of sides of the polygons involved adds up to one-half (Figure 5, right). The last identity, showing an isomorphism between angular and fractional relations, is not a coincidence but rather a special case of the general relationship that enables the edge-to-edge tessellation with three regular polygons. Indeed, by setting  $\alpha_1$ ,  $\alpha_2$ , and  $\alpha_3$  to be angles of three regular polygons with, respectively,  $n_1$ ,  $n_2$ , and  $n_3$  sides,  $n_1 > n_2 > n_3 \geq 3$ , one can show that the equality  $\alpha_1 + \alpha_2 + \alpha_3 = 180^\circ$ , implying that the three polygons enable the edge-to-edge tessellation, yields the equation  $\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} = \frac{1}{2}$ . The last equation can be interpreted in terms of partitioning of one-half into a sum of three different unit fractions. Through such interpretation, one can make a transition from the world of regular polygons to the world of unit fractions.

One can partition one-half into three distinct fractions by using fraction circles – manipulatives shaped as a sector of a whole circle. Electronic fraction circles as well as regular polygons can be constructed by using *The Geometer's Sketchpad*. The reason of establishing an isomorphism between regular polygons and fraction circles is because in the latter world one can use a system in partitioning a unit fraction into three like fractions, whereas the development of such a system in the world of regular polygons would be more complex.

By comparing basic elements of the two models – polygons and fraction circles – one can discover that internal angle of a regular polygon and the central angle of a fraction circle are supplementary angles. This observation can be easily verified through the identity  $\frac{360^\circ}{n} + \frac{n-2}{n}180^\circ = 180^\circ$ . The last identity can be seen as a new mathematical

model. Its interpretation brings about the above statement regarding supplementary angles. This model can be generalized to the form

$$\frac{2\alpha}{n} + \frac{n-2}{n}\alpha = \alpha, \quad 0 < \alpha < 180^\circ \quad (1)$$

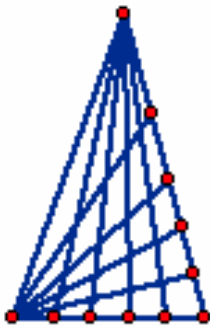
The change in the model calls for its appropriate interpretation in the language of mathematics. To this end, new contextual inquiries emerge. More specifically, the new model can be associated with what one may refer to as a double angle triangle in which two angles are in the two-to-one ratio. For example, one can be asked to prove that in such a triangle the smallest of the two angles is smaller than  $\frac{180^\circ n}{n+2}$  for all  $n > 1$ .

### Illustration 7. Modeling a Partial Difference Equation

Another, more complex interpretation of identity (1) involving angle  $\alpha$  can lead to the following inquiry into a double angle triangle. Consider a situation in which each of the angles that are in the ratio of two-to-one is divided into  $n$  congruent angles by the line segments  $l_k$  and  $l_i$  ( $k, i = 1, 2, \dots, n$ ) dropped from the corresponding vertex to the opposite side of the triangle (Figure 6). One can show that the angles  $f(k, i)$  formed at the intersection of the line segments  $l_k$  and  $l_i$  satisfy the following partial difference equation in two variables (Heins, 1941)

$$f(k, i) = f(k-1, i) + f(k, i-1), \quad f(k, 1) = \frac{(k+2)\alpha}{n}, \quad f(1, i) = \frac{(2i+1)\alpha}{n} \quad (2)$$

**Figure 6.** The case  $\alpha=35^\circ, n=5$



**Figure 7.** Spreadsheet developed from triangle

	A	B	C	D	E
1	3	4	5	6	7
2	5	9	14	20	27
3	7	16	30	50	77
4	9	25	55	105	182
5	11	36	91	196	378

In turn, Equation (2) can be modeled numerically within a spreadsheet (Figure 7), which is isomorphic to the triangle of Figure 6. The first observation that one can make about this spreadsheet is that each column consists of numbers that are one more than the sums of numbers in the previous column. Discovering other patterns within the spreadsheet enables teachers' engagement in problem-solving activities associated with summation of integer sequences with special properties. In that way, by changing model within a model one can gain experience in posing new problems and developing new mathematical concepts.

### Teachers' Voices from a Modeling Classroom

Many teachers believe that modeling activities involving manipulatives are associated with mathematics taught to young children only. It appears that age level does not limit such modeling of mathematical content. Indeed, throughout the whole K-12 curriculum, the use of manipulatives as modeling tools facilitates one's conceptual understanding and the development of problem-solving skills. This approach allows one to develop a physical model and then internalize it through appropriate mathematical interpretation. As one teacher put it: *"I think using manipulatives helps the understanding of formal mathematics because it brings you back to the base of the problem and allows you to visually see what is going on in the problem. You can take the simple problem with the manipulatives and apply it to more difficult problems."* This remark suggests that the teacher has developed appreciation of using concrete materials in modeling. Moreover, it indicates that the teacher considers a physical model as a springboard into an extended context that may include generalization.

By using tools of technology in mathematical modeling, teachers can discover isomorphic relationships among different concepts. Through such experiential approach to mathematics one can construct an "empirical situation in which [familiar] objects are differently related to one another" (Dewey, 1929, p. 86). One example of such empirical situation is the connection between edge-to-edge tessellation with regular polygons and representation of the fraction circle one-half as a sum of three fraction circles. As another teacher noted: *"I first used the fraction circles to experiment. As I tried more examples, I started to think about the process I went through. It is a great way to represent sums of unit fractions geometrically."*

Another example of that kind described in the paper is the connection (recognized, in fact, by one of the teachers) between angles of a polygon and the corresponding fraction circle. This connection can be interpreted in more general terms by making an invariant angular element of the model (the straight angle) a variable parameter. This, in turn, can lead to a new model describing complex angular relationships developed within a net of intersecting segments stemming from two vertices of the triangle of a special type (Figure 6). Reflecting on the activities, a teacher noted: *"In mathematics, it is important to experiment and formulate ideas about a solution for the problem. Once you have experimented, it is important to extend this knowledge into theoretical knowledge. When you create a theory, this theory can be applied to many different types of problems."* This note shows the teacher's understanding the value of modeling in developing theoretical knowledge from an experiment.

Furthermore, through experimentation made possible by the use of technology, one can see new directions in which mathematical explorations can proceed, discover connections among concepts, and experience mathematics as a body of knowledge that makes sense. In the words of yet another teacher: *"Throughout this class, I have also seen that there are a lot of connections in math. Learning about one topic or idea can carry over to another area. These connections help students to gain a more concrete understanding of mathematics."* The use of the word "concrete" by the teacher indicates the emergence of meaning through constructing isomorphic relationships that make mathematical concepts less abstract. The author believes that promoting sense-making modeling pedagogy in mathematics teacher education courses fosters teachers' modeling abilities and, ultimately, can positively affect how mathematics is taught in schools.

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